Turbulent Free Convection Heat Transfer From a Flat Vertical Plate to a Power Law Fluid

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It is well known that the theoretical analyses of the problem of laminar natural convection are based on the assumption that the motion is confined to a thin layer near the wall, thus implying large Grashof numbers. Therefore, the predictions of laminar free convection analyses should become increasingly accurate with increasing Grashof numbers. However, contrary results have been found experimentally, and these have been attributed to the appearance of turbulence in the flow at the top of the surface which gradually extends to cover more and more of the surface as the Grashof number increases. Turbulence may occur because the surface in question is large or the temperature difference is large.

For Newtonian fluids, turbulent natural convection has been studied by a number of workers, such as Colburn and Hougen (1930), Eckert and Jackson (1950), Bayley (1955), and Fujii (1959). However, there exists no theoretical analysis of this problem for non-Newtonian fluids, and hence the purpose of this communication is to solve the turbulent natural convection problem for viscoinelastic solutions in a manner similar to Eckert and Jackson (1950). The approximate integral method has been used to obtain the asymptotic solution for high Prandtl numbers using a similarity transformation.

THEORETICAL ANALYSIS

The plate is assumed to be flat, vertical, and semi-infinite (Figure 1), and the physical properties of the fluid (except the density in the buoyancy term) are assumed to be constant.

The integral equations for momentum and heat balances in the boundary layer can be set down in a manner similar to Eckert and Jackson (1950) as follows:

$$\frac{d}{dx}\int_0^\delta u^2\,dy = g\beta_0\int_0^\delta (T - T_\infty)\,dy - \frac{\tau_0}{\rho} \tag{1}$$

$$\frac{d}{dx}\int_0^\delta uT\,dy = -\frac{k}{\rho C_p}\left(\frac{\partial T}{\partial y}\right)_W \tag{2}$$

The boundary conditions on the velocity and temperature are as follows:

$$\begin{aligned} u(x,0) &= u(x,\delta) = 0 \\ T(x,0) &= T_W \\ T(x,\delta) &= T_\infty \end{aligned} \tag{3}$$

Dodge and Metzner (1959) have provided a Blasius type of approximate equation for the friction factor generalized Reynolds number as follows:

$$f = \frac{\alpha}{(N_{Regen})^\beta} \quad 5\times 10^3 \leq N_{Regen} \leq 10^5 \tag{4}$$

where α and β are functions of n for the case of power law fluids, and an explicit equation in f for turbulent flow of power law fluids in smooth tubes results. Following the procedure of Skelland (1967), a suitable expression for the local surface shear stress can be obtained by proper rearranging and adapting the equations to flow over a smooth flat plate at zero incidence in a manner analogous to that used by Eckert and Jackson (1950) in the Newtonian case as

$$\tau_o = \Omega\rho^{1-\beta}\,\gamma_1^\beta\,\delta^{-\beta n}\,\Lambda^{2-\beta(2-n)} \tag{5}$$

where

$$\Omega = \frac{\alpha(0.817)^{2-\beta(2-n)}}{2^{\beta n+1}} \tag{6}$$

and

$$\gamma_1 = 8^{n-1}K\left(\frac{3n+1}{4n}\right)^n \tag{7}$$

For the Newtonian case

$$n = 1,\quad \beta = 0.25,\quad \Omega = 0.02332,\quad \gamma_1 = \mu$$

$$\tau_{oN} = 0.02332\,\rho\,\Lambda^2\left(\frac{\mu}{\rho\delta\Lambda}\right)^{1/4} \tag{8}$$

Now, as introduced by Skelland (1967), an effective viscosity can be written which makes the above Newtonian equation fit the turbulent power law conditions described by Equation (5):

$$\mu_{eff} = \left(\frac{\Omega}{0.02332}\right)^4 \gamma_1^{4\beta}\,\rho^{1-4\beta}\,\delta^{1-4\beta n}\,\Lambda^{1-4\beta(2-n)} \tag{9}$$

The Colburn's analogy between heat and momentum transfer may now be applied to the turbulent flow over the flat plate as

$$\frac{h_W}{\rho C_p\Lambda}\left(\frac{C_p\,\mu_{eff}}{k}\right)^{2/3} = \frac{\tau_o}{\rho\Lambda^2} \tag{10}$$

thus yielding

$$h_W = \Omega\,C_p\,\rho^{1-\beta}\,\gamma_1^\beta\,\delta^{-\beta n}\,\Lambda^{1-\beta(2-n)}\left(\frac{C_p\,\mu_{eff}}{k}\right)^{-2/3} \tag{11}$$

With the assumption made by Eckert and Jackson

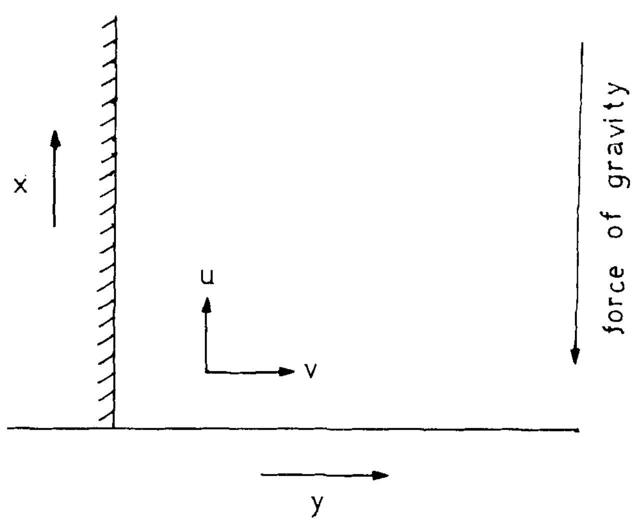

Fig. 1. Schematic diagram of flow past a semi-infinite vertical flat plate.

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(1950) that close to the wall the relationships connecting wall shear stress and heat flow with temperatures and velocities in this range are the same for forced flow and free convection flow, the following can be used:

$$-k\left(\frac{\partial T}{\partial y}\right)_W = q_W = h_W\,(T_W - T_\infty) \tag{12}$$

Using Equations (5), (11), and (12), the simplified forms of Equations (1) and (2) can now be obtained as follows:

$$\frac{d}{dx}\int_0^\delta u^2\,dy = g\beta_o \int_0^\delta (T - T_\infty)dy - \Omega\rho^{-\beta}\,\gamma_1^{\beta}\,\delta^{-\beta n}\,\Lambda^{2-\beta(2-n)} \tag{13}$$

$$\frac{d}{dx}\int_0^\delta uT\,dy = (T_W - T_\infty)\,\Omega\rho^{-\beta}\,\gamma_1^{\beta}\,\delta^{-\beta n}\,\Lambda^{1-\beta(2-n)}\left(\frac{C_p\,\mu_{\text{eff}}}{k}\right)^{-2/3} \tag{14}$$

An order of magnitude analysis of Equations (13) and (14) can be carried out using $u \sim 0(U_c)$, $x \sim 0(l_c)$, $y \sim 0(\delta)$ and $\Lambda \sim 0(U_c)$. It can be then easily seen that for large values of a characteristic Prandtl number N_{Pr_c} defined as

$$N_{Pr_c} = \frac{C_p}{k}\,\gamma_1^{4\beta}\,\rho^{1-4\beta}\,l_c^{\frac{3-4\beta(2+n)}{2}}\,[g\beta_o\,(T_W - T_\infty)]^{\frac{1-4\beta(2-n)}{2}} \tag{15}$$

for the constant temperature plate, the inertial terms in the momentum equation are negligible in comparison to the other terms on the right-hand side of Equation (13). This assumption is congruous with the fact that non-Newtonian fluids have high consistencies.

A characteristic Grashof number can now be defined by taking the ratio of the buoyancy force to the viscous force as

$$N_{Gr_c} = \gamma_1^{-8\beta}\,\rho^{8\beta}\,l_c^{4\beta(2+n)}\,[g\beta_o\,(T_W - T_\infty)]^{4\beta(2-n)} \tag{16}$$

As there exists no characteristic length for the external flow past the semi-infinite plate under consideration, the method of Hellums and Churchill (1964) is used to choose l_c such that $N_{Gr_c} = 1$. Thus

$$l_c = \left\{\frac{\gamma_1^2}{\rho^2\,[g\,\beta_o\,(T_W - T_\infty)]^{2-n}}\right\}^{\frac{1}{2+n}} \tag{17}$$

A characteristic velocity U_c is now defined as follows:

$$U_c = \left\{\frac{\gamma_1\,[g\beta_o\,(T_W - T_\infty)]^n}{\rho}\right\}^{\frac{1}{2+n}} \tag{18}$$

The nondimensional variables are now defined as

$$x_1 = \frac{x}{l_c};\quad y_1 = \frac{y}{l_c};\quad \delta_1 = \frac{\delta}{l_c}$$
$$u_1 = \frac{u}{U_c};\quad A = \frac{\Lambda}{U_c};\quad \theta = \frac{T - T_\infty}{T_W - T_\infty} \tag{19}$$

The nondimensional forms of Equation (13) (on neglecting inertia) and Equation (14) can now be written as

$$0 = \int_0^{\delta_1} \theta\,dy_1 - \Omega\,\delta_1^{-\beta n}\,A^{2-\beta(2-n)} \tag{20}$$

$$\frac{d}{dx_1}\int_0^{\delta_1} u_1\,\theta\,dy_1 = \Omega\left(\frac{0.02332}{\Omega}\right)^{\frac{8}{3}} N_{Gr_x}^{\frac{3-4\beta(2+n)}{12\beta(2+n)}}\,N_{Pr_x}^{-\frac{2}{3}}\,\delta_1^{\frac{5\beta n-2}{3}}\,A^{\frac{1+5\beta(2-n)}{3}} \tag{21}$$

where N_{Gr_x} is the local distance based Grashof number defined as

$$N_{Gr_x} = \gamma_1^{-8\beta}\,\rho^{8\beta}\,x^{4\beta(2+n)}\,[g\beta_o(T_W - T_\infty)]^{4\beta(2-n)} \tag{22}$$

and N_{Pr_x} is the local distance based Prandtl number defined as

$$N_{Pr_x} = \frac{C_p}{k}\,\gamma_1^{4\beta}\,\rho^{1-4\beta}\,x^{\frac{3-4\beta(2+n)}{2}}\,[g\beta_o(T_W - T_\infty)]^{\frac{1-4\beta(2-n)}{2}} \tag{23}$$

Note that $N_{Gr_x}^{\frac{3-4\beta(2+n)}{12\beta(2+n)}}\,N_{Pr_x}^{-2/3}$ is independent of x and will be treated like a constant during the following analysis.

Equations (20) and (21) are now solved for the following boundary conditions:

$$u_1(x_1, 0) = u_1(x_1, \delta_1) = 0$$
$$\theta(x_1, 0) = 1 \tag{24}$$
$$\theta(x_1, \delta_1) = 0$$

In a manner similar to Eckert and Jackson (1950), expressions for u_1 and θ are to be sought. They noted that in forced convection the equations $u = \Lambda(y/\delta)^{1/7}$ and $(T - T_\infty) = (T_W - T_\infty)\,\{1 - (y/\delta)^{1/7}\}$ hold considerably well. Observing temperature and velocity distributions obtained experimentally by Griffiths and Davis (1922), they concluded that the temperature equation fitted the free convection experimental data reasonably well, while, of course, the velocity profile showed a different trend owing to the fact that in free convection the velocity is zero both at the solid surface and remote from it. They, however, found that the equation $u = \Lambda(y/\delta)^{1/7}(1 - y/\delta)^4$ fitted the shape quite well.

For power law fluids, the velocity profile for turbulent forced convection flows can be assumed as $u = \Lambda(y/\delta)^q$, where $q = \beta n/[2 - \beta(2 - n)]$ as given by Skelland (1967). In the present free convection case, the velocity and temperature profiles will be assumed by analogous arguments to those of Eckert and Jackson (1950), making use of the forced convection expression for power law fluids as given by Skelland (1967). Thus, the dimensionless temperature and velocity profiles which are assumed to fit the turbulent free convection flow of power law fluids are

$$\theta(\eta) = 1 - \eta^q \tag{25}$$

$$u_1(\eta) = A\eta^q\,(1 - \eta)^4 \tag{26}$$

where

$$\eta = \frac{y_1}{\delta_1} \tag{27}$$

and

$$q = \frac{\beta n}{2 - \beta(2-n)} \tag{28}$$

Substituting Equations (25) and (26) in Equations (20) and (21), and appropriately rearranging the terms, we obtain

$$0 = C_1\,\delta_1 - \Omega\,\delta_1^{-\beta n}\,A^{2-\beta(2-n)} \tag{29}$$

$$C_2 \frac{d}{dx_1}(\delta_1 A) = \Omega\left(\frac{0.02332}{\Omega}\right)^{\frac{8}{3}} N_{Gr_x}^{\frac{3-4\beta(2+n)}{12\beta(2+n)}}\, N_{Pr_x}^{-\frac{2}{3}}\, \delta_1^{\frac{5\beta n-2}{3}}\, A^{\frac{1+5\beta(2-n)}{3}} \tag{30}$$

where

$$C_1 = \frac{q}{q+1} \tag{31}$$

$$C_2 = \frac{3}{q+1} - \frac{2}{q+2} + \frac{6}{q+3} - \frac{4}{q+4} + \frac{1}{q+5} - \frac{1}{2q+1} - \frac{6}{2q+3} - \frac{1}{2q+5} \tag{32}$$

For a similarity search, the following forms of δ_1 and A are assumed:

$$\delta_1 = B_1\,x_1^{r} \tag{33}$$

$$A = B_2\,x_1^{s} \tag{34}$$

Substituting these into Equations (29) and (30) and equating the powers of x_1 for the equations to be valid for any x_1, we find that

$$r = \frac{3[2-\beta(2-n)]}{2[6-\beta(10-n)]} \tag{35}$$

$$s = \frac{3[(1+\beta n)]}{2[6-\beta(10-n)]} \tag{36}$$

thus giving the conditions for similarity. Note that r and s reduce correctly to their respective Newtonian values of 7/10 and 1/2 as obtained by Eckert and Jackson (1950).

The simplified forms of Equations (29) and (30) on substitution of (33), (34), (35), and (36) are

$$C_1 B_1^{1+\beta n} = \Omega\, B_2^{2-\beta(2-n)} \tag{37}$$

$$B_1^{\frac{5(1-\beta n)}{3}}\, B_2^{\frac{2-5\beta(2-n)}{3}} = \frac{\Omega}{C_2}\left(\frac{0.02332}{\Omega}\right)^{8/3} \left\{\frac{2[6-\beta(10-n)]}{9-6\beta(1-n)}\right\} N_{Gr_x}^{\frac{3-4\beta(2+n)}{12\beta(2+n)}}\, N_{Pr_x}^{-\frac{2}{3}} \tag{38}$$

Thus, solving (37) and (38), we get

$$B_1 = \left(\frac{\Omega}{C_1}\right)^{\frac{2-5\beta(2-n)}{2[6-\beta(10-n)]}} \left(\frac{\Omega}{C_2}\right)^{\frac{3[2-\beta(2-n)]}{2[6-\beta(10-n)]}} \left(\frac{0.02332}{\Omega}\right)^{\frac{4[2-\beta(2-n)]}{[6-\beta(10-n)]}} \left\{\frac{2[6-\beta(10-n)]}{9-6\beta(1-n)}\right\}^{\frac{3[2-\beta(2-n)]}{2[6-\beta(10-n)]}} N_{Gr_x}^{\frac{[3-4\beta(2+n)]\,[2-\beta(2-n)]}{[8\beta(2+n)]\,[6-\beta(10-n)]}}\, N_{Pr_x}^{-\frac{[2-\beta(2-n)]}{[6-\beta(10-n)]}} \tag{39}$$

and

$$B_2 = \left(\frac{C_1}{\Omega}\right)^{\frac{1}{[2-\beta(2-n)]}} B_1^{\frac{[1+\beta n]}{[2-\beta(2-n)]}} \tag{40}$$

Now, the local Nusselt number is defined as

$$N_{Nu_x} = \frac{h_W}{k} \tag{41a}$$

$$= \Omega\left(\frac{0.02332}{\Omega}\right)^{8/3} \delta_1^{\frac{5\beta n-2}{3}}\, A^{\frac{1+5\beta(2-n)}{3}}\, N_{Gr_x}^{\frac{3+4\beta(2+n)}{24\beta(2+n)}}\, N_{Pr_x}^{\frac{1}{3}} \tag{41b}$$

Substituting appropriately and simplifying, we get

$$N_{Nu_x} = (0.02332)^{\frac{4[3-2\beta(1-n)]}{[6-\beta(10-n)]}} (\Omega)^{-\frac{9}{[6-\beta(10-n)]}} (C_1)^{\frac{1}{[2-\beta(2-n)]}} (C_2)^{\frac{[3-2\beta(7+2n)]}{2[6-\beta(10-n)]}} (C_1)^{\frac{[2-5\beta(2-n)]\,[-3+2\beta(1-n)]}{2[6-\beta(10-n)]\,[2-\beta(2-n)]}} \left\{\frac{2[6-\beta(10-n)]}{[9-6\beta(1-n)]}\right\}^{\frac{[-3+2\beta(7+2n)]}{2[6-\beta(10-n)]}} N_{Gr_x}^{\frac{[21-4\beta(8+n)]}{8[6-\beta(10-n)]}}\, N_{Pr_x}^{\frac{[3-\beta(8+n)]}{[6-\beta(10-n)]}} \tag{42}$$

Thus, Equation (42) can be expressed as

$$N_{Nu_x} = C\, N_{Gr_x}{}^{a}\, N_{Pr_x}{}^{b} \tag{43}$$

with the appropriate definitions of C, a, and b, the values of which for varying n are tabulated in Table 1. With increasing pseudoplasticity, the coefficient C and exponent a increase continuously, while the exponent b decreases. But the trend of Nusselt number cannot be easily ascertained owing to the unknown Grashof and Prandtl number changes with increasing pseudoplasticity. However, Equation (42) could prove very useful in the design of turbulent natural convection processes in non-Newtonian fluids.

With the assumption that the boundary layer is turbulent over the whole of the plate, the average Nusselt number can be easily written by taking an integrated average over the length of the plate L as

$$N_{Nu_{av}} = (0.02332)^{\frac{4[3-2\beta(1-n)]}{[6-\beta(10-n)]}} (\Omega)^{-\frac{9}{[6-\beta(10-n)]}}$$

TABLE 1

No.	n	α	β	C	a	b
1	1.0	0.0790	0.250	0.0402	0.400	0.200
2	0.9	0.0770	0.255	0.0428	0.405	0.199
3	0.8	0.0760	0.263	0.0443	0.410	0.192
4	0.7	0.0752	0.270	0.0450	0.416	0.187
5	0.6	0.0740	0.281	0.0464	0.422	0.174
6	0.5	0.0723	0.290	0.0477	0.429	0.165
7	0.4	0.0710	0.307	0.0483	0.438	0.138
8	0.3	0.0683	0.325	0.0497	0.448	0.106
9	0.2	0.0646	0.349	0.0501	0.463	0.054

$$(C_1)^{\frac{1}{[2-\beta(2-n)]}} (C_2)^{\frac{[3-2\beta(7+2n)]}{2[6-\beta(10-n)]}} (C_1)^{\frac{[2-5\beta(2-n)][-3+2\beta(1-n)]}{2[6-\beta(10-n)][2-\beta(2-n)]}} \left\{ \frac{2[6-\beta(10-n)]}{[9-6\beta(1-n)]} \right\}^{\frac{[9-6\beta(1-n)]}{2[6-\beta(10-n)]}} N_{Gr_L}^{\frac{[21-4\beta(8+n)]}{8[6-\beta(10-n)]}} N_{Pr_L}^{\frac{[3-\beta(8+n)]}{[6-\beta(10-n)]}} \quad (44)$$

In reality, of course, the boundary layer is initially laminar and becomes turbulent only at a certain distance from the leading edge of the plate. However, Equation (44) could be expected to predict true average Nusselt number values only at Grashof numbers which are so large that the extent of the laminar boundary layer at the lower edge of the plate is small compared with the total length L of the plate. This limit for the Grashof number has been suggested to be around 10^{10} for Newtonian fluids by Eckert and Jackson (1950) and could be assumed to be around the same for the power law equivalent of the Grashof number in the present analysis.

Finally, it is worth mentioning that although the development in the present analysis is closely based on that of Eckert and Jackson (1950), Equations (43) and (44) differ in structure from their expressions for the Newtonian case owing to the extra assumption of high Prandtl number made herein and could be matched only when a similar assumption is made in the final forms of their local and average Nusselt numbers.

ACKNOWLEDGMENT

The financial support provided by the British Gas Corporation during the tenure of this work is greatly appreciated.

NOTATION

a = exponent of Grashof number in Equation (43)
A = dimensionless velocity term defined in Equation (19)
b = exponent of Prandtl number in Equation (43)
B_1 = coefficient in Equation (33)
B_2 = coefficient in Equation (34)
C = coefficient in Equation (43)
C_1 = function of β, n as defined by Equation (31)
C_2 = function of β, n as defined by Equation (32)
C_p = specific heat per unit mass
g = acceleration due to gravity
h_W = coefficient of heat transfer at the wall defined by Equation (11)
k = thermal conductivity
K = consistency index for a power law fluid
l_c = characteristic length
n = pseudoplasticity index for a power law fluid
N_{Gr_c} = characteristic generalized Grashof number for a power law fluid defined by Equation (16)
N_{Gr_x} = generalized local Grashof number defined by Equation (22)
N_{Pr_c} = characteristic generalized Prandtl number for a power law fluid defined by Equation (15)
N_{Pr_x} = generalized local Prandtl number defined by Equation (23)
$N_{Re_{gen}}$ = generalized Reynolds number
N_{Nu_x} = local Nusselt number defined by Equation (41*a*)
q = function of β, n as defined by Equation (28)
q_W = heat flux at the wall defined by Equation (12)
r = function of β, n as defined by Equation (35)
s = function of β, n as defined by Equation (36)
T = temperature
T_W = temperature of the wall
T_∞ = temperature of the bulk of the fluid
u = velocity component along the x coordinate
u_1 = dimensionless velocity component defined in Equation (19)
U_c = characteristic velocity defined by Equation (18)
x = distance along the plate from the leading edge
x_1 = dimensionless distance defined in Equation (19)
y = distance normal to the plate
y_1 = dimensionless distance defined in Equation (19)

Greek Letters

α, β = dimensionless functions of n appearing in Equation (4)
β_o = expansion coefficient of the fluid
γ_1 = coefficient defined by Equation (7)
δ = boundary layer thickness
δ_1 = dimensionless boundary layer thickness defined in Equation (19)
η = similarity variable defined by Equation (27)
Λ = velocity component appearing in Equation (5)
θ = dimensionless temperature difference defined in Equation (19)
μ_{eff} = effective viscosity defined by Equation (9)
ρ = density of the fluid
τ_o = shear stress at the wall defined by Equation (5)
τ_{oN} = shear stress at the wall for a Newtonian fluid given by Equation (8)
Ω = coefficient defined by Equation (6)

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Manuscript received May 24, 1977; revision received July 22, and accepted August 11, 1977.